Problem 2.

(1) Decide, whether a set $[x, y, u, v] \in \mathbb{R}^4$ which satisfies

$$\begin{aligned} xe^{u+v} + 2(u+v)y &= 1, \\ ye^{u-v} - \frac{u}{1+v} &= 2x \end{aligned}$$

can be described on some neighborhood of $[1, 2, 0, 0]$ as a graph of a continuously differentiable function $[g, h](x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ ($[g, h](x, y) = [u, v]$) which is defined on some neighborhood of $[1, 2]$ satisfying $g(1, 2) = h(1, 2) = 0$.

(2) Compute the partial derivative of function g with respect to y at $[1, 2]$.

If you use the implicit function theorem, then verify its conditions.

Solution

(1) We are going to verify conditions of the implicit function theorem for equation

$$[G, H](x, y, u, v) = \left(xe^{u+v} + 2(u+v)y - 1, ye^{u-v} - \frac{u}{1+v} - 2x \right) = [0, 0]$$

and point $[1, 2, 0, 0]$.

- Function G and $ye^{u-v} - 2x$ are continuously differentiable on $\mathbb{R}^4$. Function $-\frac{u}{1+v}$ is continuously differentiable everywhere except points, where $v = -1$. Thus, $F \in \mathcal{C}^1(B([1, 2, 0, 0], 1))$.
- $F(1, 2, 0, 0) = [0, 0]$,
- $$\begin{vmatrix} \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \\ \frac{\partial H}{\partial u} & \frac{\partial H}{\partial v} \end{vmatrix} (1, 2, 0, 0) = \begin{vmatrix} 5 & 5 \\ 1 & -2 \end{vmatrix} = -15 \neq 0.$$

  Thus, g and h exist and belong to $\mathcal{C}^1$.

(2) Vector $\begin{pmatrix} \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial y} \end{pmatrix} (1, 2)$ is the solution of system of linear equations

$$\left(\begin{array}{cc|c} \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} & -\frac{\partial G}{\partial y} \\ \frac{\partial H}{\partial u} & \frac{\partial H}{\partial v} & -\frac{\partial H}{\partial y} \end{array} \right) (1, 2, 0, 0) = \left(\begin{array}{cc|c} 5 & 5 & 0 \\ 1 & -2 & -1 \end{array} \right).$$

Thus $\frac{\partial g}{\partial y} = -\frac{1}{3}$.